In memory of Professor Zhilin

On the 2D models of plates and shells including the transversal shear

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The 2D Kirchhoff–Love (KL) theory and the Timoshenko–Reissner (TR) theory for thin shells made of the transversal isotropic homogeneous material are discussed. For the cyclic-symmetric deformations of shells of revolution the asymptotic analysis of strain–stress states is fulfilled. Two simple linear problems for double-periodic deformations of plates are studied basing on the exact 3D theory and on the 2D approximate theories. From these problems it follows that the KL theory is asymptotically correct because it gives the first term of asymptotic expansion of the 3D solution. The TR theory is asymptotically incorrect. It also gives correctly the first term and incorrectly gives the second term. But if the transversal shear module is comparatively small then this theory gives the main part of the second term. The case of the extremely small shear module is discussed. As an example the multi-layered plate with the alternating hard and soft isotropic layers is studied.

1 Introduction

In this paper the 2D Kirchhoff–Love (KL) theory and the Timoshenko–Reissner (TR) theory for thin shells made of the transversal isotropic homogeneous material are discussed. The KL theory is the simplest 2D theory of shells. It may be delivered from the 3D equations of the theory of elasticity as by the hypothesis of straight normal so by asymptotic expansions of solution in powers of a small parameter $h_* = h/R$ which is equal to the relative shell thickness (here h is the shell thickness and R is the typical radius of curvature). The first estimation of the KL theory error Δ for isotropic shell $\Delta \sim h_*$ is obtained in [1].

The error of the KL theory depends on the stress-strain state namely on the index of its variation t (see (10)). The more exact relation for Δ namely $\Delta \sim h_*^{2-2t}$, $0 \leq t < 1$, is delivered in [2]. The stress state of a shell as the 3D elastic body consists of three parts:

(i) of the internal (or of the main) state for the points which are far from the shell edges and from the special points (the concentrated loads, points of discontinuity and so on),

(ii) of the edge effect state which as a rule occupy the zone with the width of the order $\sqrt{Rh}$, and the index t of this state variation is equal to $t = 1/2$,

(iii) of the boundary layer which occupy the zone with the width of the order h, and the index t of this state variation is equal to $t = 1$.

The first two states (i) and (ii) may be described approximately by the 2D shell theories, and for this states the mentioned above errors Δ for the KL theory are valid. The third zone (iii) is essentially 3D and the construction of solution in this zone and its connection with the solutions of (i) and (ii) types is a special problem (see [3], [4]).

The 2D TR theory [5] is to be more exact than the KL theory. The advantages of the TR theory compared with the KL theory are the following. The equations of this theory are of the 10th differential order while the KL theory has the 8th order. That is why the boundary conditions in the TR theory are formulated more naturally because to each of 5 generalized degrees of freedom of the normal element (3 displacements and 2 angles of rotation) the separate boundary condition corresponds. In dynamics the TR theory leads to the hyperbolic system of equations [6] and it is convenient to describe the waves propagation.

In the other side for the most statical problems for shells made of isotropic homogeneous material the correction which gives the TR theory compared with the KL theory is inessential. More over the TR theory is asymptotically conflicting.

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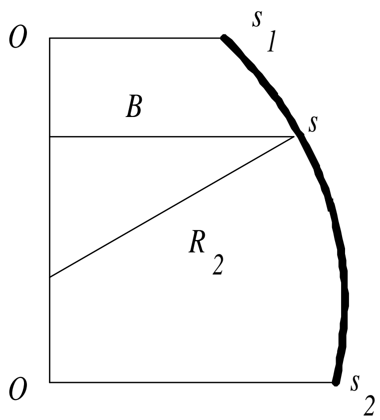

Fig. 1 The shell of revolution.

Some of solutions of the corresponding system have the index of variation $t = 1$. The very short length $L \sim h$ of the deformation picture corresponds to these solutions and the theory is doubtful because the 2D shell theories are based on the assumption that $L >> h$. Nevertheless a lot of statical and dynamical solutions based on the TR theory are obtained and the influence of the more detailed boundary conditions than it is possible at the KL theory are investigated (see papers [7], [8], [9], [10] and others).

There exist a lot of more general 2D shell theories than the KL theory and the TR theory. The additional degree of freedom connected with the normal element rotation around itself is introduced in the work [11]. The corresponding system is of the 12th differential order. In the works [12], [13], [14] the new approach to the shell theory construction is proposed. By using the Cosserat's ideas the general form of the constitutive relations is delivered from the lows of thermo-dynamics and then the necessary constants are obtained from some test problems which have the exact 3D solution. The resulting system again is of the 12th differential order. In the book [15] by using hypotheses about the stresses and strains distribution in the thickness direction the system of the 14th order is proposed. This system allows us to satisfy boundary conditions on all shell surfaces and it is acceptable also for anisotropic shells. Some theories of anisotropic shells are contained in the book [16].

Let us return to the TR model for the transversal isotropic shell the shear module of which in the transversal direction is much smaller than the module in the tangential directions. We put $G_{33}/E \sim h_*^\delta$ and suppose that $\delta > 0$. It occurs that for such shell the using of the TR theory is not only correct but it is more precise than the KL theory.

In this paper for the cyclic-symmetric deformations of shells of revolution the asymptotic analysis of strain–stress states based on the equations of the TR theory is fulfilled. It is shown that in the case $\delta > 0$ the indexes of variation of solutions $t < 1$ hence the 2D shell theory is acceptable.

Two simple linear problems for double-periodic deformations of plates are studied basing on the exact 3D theory and on the 2D approximate theories. From these problems it follows that the KL theory is asymptotically correct because it gives the first term of asymptotic expansion of the 3D solution. The TR theory is asymptotically incorrect. It also gives correctly the first term and incorrectly gives the second term. But if the transversal shear module is comparatively small ($\delta > 0$) then this theory gives the main part of the second term. As an example the multi-layered plate with the alternating hard and soft isotropic layers is studied. The equivalent transversal isotropic plate is constructed. For this plate the 2D and the 3D solutions are compared and it is shown that in this problem the TR theory is more exact than the KL theory.

2 The 2D equations for a shell of revolution and their asymptotic analysis

For simplicity we study the cyclic-symmetric deformations of shells of revolution (Fig. 1) for which the transversal deflection is

$$w(s, \varphi) = w(s) \cos m\varphi, \tag{1}$$

where s is the length of the generatrix arc, φ is the angle in the circumferential direction, and m is the number of waves in this direction. The other unknown functions are also taken in the form $x(s) \cos m\varphi$ or $x(s) \sin m\varphi$.

We have the following expressions describing the mid-surface small deformations

$$\begin{aligned}
&\varepsilon_1 = u_1' - \frac{w}{R_1}, \quad \varepsilon_2 = \frac{B'}{B}u_1 + \frac{m}{B}u_2 - \frac{w}{R_2}, \quad \omega = B\left(\frac{u_2}{B}\right)' - \frac{m}{B}u_1, \\
&\gamma_1 = -w' - \frac{u_1}{R_1}, \quad \gamma_2 = \frac{mw}{B} - \frac{u_2}{R_2}, \qquad (\,)' \equiv \frac{d(\,)}{ds}, \\
&\kappa_1 = -\gamma_1', \quad \kappa_2 = -\frac{m\gamma_2}{B} - \frac{B'}{B}\gamma_1, \quad \tau = \frac{m\gamma_1}{B} + \frac{B'}{B}\gamma_2 + \frac{u_2'}{R_2},
\end{aligned} \tag{2}$$

where $\varepsilon_1,\ \varepsilon_2,\ \omega$ are the tangential deformations, $\gamma_1,\ \gamma_2$ are the angles of rotation of the mid-surface normal, $\kappa_1,\ \kappa_2,\ \tau$ are the bending-torsion deformations of the mid-surface. In the KL model the angles γ_1 and γ_2 are coincide with the angles of the normal element rotation. And in the TR model angles of the normal element rotation $\varphi_1,\ \varphi_2$ are the independent variables and

$$\varphi_1 = \gamma_1 + \delta_1, \qquad \varphi_2 = \gamma_2 + \delta_2, \tag{3}$$

where $\delta_1,\ \delta_2$ are the shear angles.

The equilibrium equations are the same as for the KL model so for the TR model

$$\begin{aligned}
&T_1' + \frac{B'}{B}(T_1 - T_2) + \frac{mS}{B} - \frac{Q_1}{R_1} + q_1 = 0, \\
&S' + \frac{2B'}{B}S - \frac{m}{B}T_2 - \frac{Q_2}{R_2} + q_2 = 0, \\
&\frac{(BQ_1)'}{B} + \frac{mQ_2}{B} + \frac{T_1}{R_1} + \frac{T_2}{R_2} + q_n = 0, \\
&M_1' + \frac{B'}{B}(M_1 - M_2) + \frac{mH}{B} + Q_1 + m_1 = 0, \\
&H' + \frac{2B'}{B}H - \frac{m}{B}M_2 + Q_2 + m_2 = 0.
\end{aligned} \tag{4}$$

where $T_1,\ T_2,\ S$ are the tangential stress-resultants, $M_1,\ M_2,\ H$ are the stress-couples, $Q_1,\ Q_2$ are the shear stress-resultants, $q_1,\ q_2,\ q_n$ and $m_1,\ m_2$ are the densities of the external forces and moments, $B(s)$ is the distance from the point in the mid-surface to the axis of rotation, $R_1(s),\ R_2(s)$ are the main radii of the mid-surface curvature.

The constitutive equations for the transversal isotropic material are

$$\begin{aligned}
&T_1 = K(\varepsilon_1 + \nu\varepsilon_2), \quad \{1, 2, cycle\}, \quad S = \frac{K(1-\nu)}{2}\omega, \quad K = \frac{Eh}{1-\nu^2}, \\
&M_1 = D(\kappa_1 + \nu\kappa_2), \quad \{1, 2, cycle\}, \quad H = D(1-\nu)\tau, \quad D = \frac{Eh^2}{12(1-\nu^2)}.
\end{aligned} \tag{5}$$

where $E,\ \nu,\ h$ are the Young's modulus, the Poisson's ratio in the tangential directions, and the shell thickness respectively.

For the KL model the values $\kappa_1,\ \kappa_2,\ \tau$ are given in (2), the shear stress-resultants $Q_1,\ Q_2$ may be found from system (4), and this system is of the 8th order.

For the TR model the values $\kappa_1,\ \kappa_2,\ \tau$ in (5) are

$$\kappa_1 = -\varphi_1', \quad \kappa_2 = -\frac{m\varphi_2}{B} - \frac{B'}{B}\varphi_1, \quad 2\tau = -B\left(\frac{\varphi_2}{B}\right)' + \frac{m}{B}\varphi_1 \tag{6}$$

and the shear stress-resultants are

$$Q_1 = \Gamma h\delta_1, \qquad Q_2 = \Gamma h\delta_2, \qquad \Gamma = \kappa G_{13}, \tag{7}$$

where G_{13} is the shear modulus in the transversal direction.

The dimensionless coefficient κ depends on the shear stresses $\sigma_{13}(z),\ \sigma_{23}(z)$ distribution in the normal direction. If these functions are paraboloidal then $\kappa = 5/6$ (see also section 4). The optimal value κ depends on the problem under consideration. In the case of the waves propagation the root of equation

$$(1-\kappa)\left(1 - \frac{(1-2\nu)\kappa}{2(1-\nu)}\right) = \left(1 - \frac{\kappa}{2}\right)^2 \tag{8}$$

is recommended as κ [17]. The value $\kappa = 5/(6-\nu)$ is obtained in the paper [13], the value $\kappa = 20/(24-3\nu)$ is obtained in the book [18].

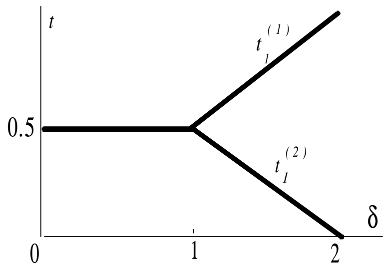

Fig. 2 Indexes of variation.

2.1 Asymptotic analysis

Let the mid-surface is bounded by two parallels $s = s_1$ and $s = s_2$. As the main small parameter we use the relative shell thickness $h_* = h/R$ where R is the typical linear dimension (for example the radius of curvature). We suppose that the transversal shear modulus is comparatively small and the integer number m changes in the wide limits

$$\Gamma/E \sim h_*^{\delta}, \quad \delta > 0; \qquad m \sim h_*^{-t_2}, \quad 0 \leq t_2 < 1. \tag{9}$$

Value t_2 is the so called index of variation in the circumferential direction. The index of variation t for function $F(z, h_*)$ in the direction z is introduced by A.L.Goldenweiser [19]

$$\frac{dF}{dz} \sim h_*^{-t} F. \tag{10}$$

The index of variation is connected with the typical length L_* of the deformation picture by relation $L_* \sim R h_*^t$.

At $m = 0$ system (4) separates into three equations (the first, the third, and the forth equations) describing the axisymmetric deformation and into the rest two equations describing the torsion.

2.2 The axisymmetric stress-strain state

The axisymmetric stress-strain state in statics is a sum of the momentless (membranous) state and the edge effect. The first of them satisfy to equations

$$T_1' + \frac{B'}{B}(T_1 - T_2) + \frac{mS}{B} + q_1 = 0, \quad \frac{T_1}{R_1} + \frac{T_2}{R_2} + q_n = 0. \tag{11}$$

Let us construct the edge effect near the edge $s = s_1$ and put $R = R_2(s_1)$. Then the edge effect is described by the approximate equation

$$D\frac{d^4 w}{ds^4} - \frac{DE}{\Gamma R^2}\frac{d^2 w}{ds^2} + \frac{Eh}{R^2} w = 0. \tag{12}$$

In this equation the asymptotically small summands are omitted and the coefficients are supposed approximately to be constant. The solution of this equation is

$$w = C_1 \exp\{\frac{p^{(1)}\, s}{R}\} + C_2 \exp\{\frac{p^{(2)}\, s}{R}\}, \qquad \mu_0^4 = \frac{h^2}{12(1-\nu^2)R^2}, \quad Re\{p^{(i)}\} < 0. \tag{13}$$

The parameters $p^{(1)}$ and $p^{(2)}$ satisfy to equation

$$\mu_0^4 p^4 - \eta_0 p^2 + 1 = 0, \qquad \eta_0 = \frac{\mu_0^4 E}{\Gamma} \sim h_*^{2-\delta}, \quad \mu_0 \sim h_*^{1/2}. \tag{14}$$

Here η_0 is the shear parameter. At $\delta \leq 1$ the index of variation for the both summands in (14) is equal to $t_1 = 1/2$, and it is the same as for the simple edge effect [19]. At $1 < \delta < 2$ the orders of roots $p^{(1)}$ and $p^{(2)}$ of equation (14) are various. The corresponding indexes of variation $t_1^{(1)} = \delta/2$ and $t_1^{(2)} = (2-\delta)/2$ are shown in Fig. 2. For the extremely small shear modulus $\delta \geq 2$ the edge effect degenerates, and this case is not studied here.

The second and the fifth equations (4) describe the shell *torsion*. Here stress-strain state consists of the membranous part satisfying to equation

$$S' + \frac{2B'}{B}S - \frac{m}{b}T_2 + q_2 = 0, \tag{15}$$

and of the edge effect

$$\frac{D(1-\nu)}{2}\varphi_2'' - \Gamma h\varphi_2 = 0. \tag{16}$$

This edge effect is absent at the KL model and its index of variation is equal to $t_1 = (2-\delta)/2$.

2.3 The cyclic-symmetric stress-strain state ($m \neq 0$)

At first we present the simplified system (4) which is similar to the Mushtary–Donnell–Vlasov's system of shallow shells and it is acceptable to describe the deformed states with the positive index of variation [20]. For this aim we suppose that $q_1 = q_2 = m_1 = m_2 = 0,\ q_n \neq 0$ and introduce the stress function Φ by relations

$$T_1 = -\frac{m^2\Phi}{B^2} + \frac{B'}{B}\frac{d\Phi}{ds}, \quad T_2 = \frac{d^2\Phi}{ds^2}, \quad S = \frac{m}{B}\frac{d\Phi}{ds} - \frac{mB'\Phi}{B^2}. \tag{17}$$

Then instead of angles φ_1 and φ_2 we introduce two new unknown functions Ψ and Θ by relations

$$\varphi_1 = -\frac{\partial\Psi}{ds} + \frac{m}{B}\Theta, \quad \varphi_2 = \frac{m}{B}\Psi - \frac{d\Theta}{ds}. \tag{18}$$

Then after omitting some small terms we get the following system

$$\frac{D(1-\nu)}{2}\Delta\Theta - \Gamma h\Theta = 0; \tag{19}$$

$$\begin{aligned}
&(Eh)^{-1}\Delta\Delta\Phi + \Delta_R w = 0,\\
&\Gamma h\left(\Delta w - \Delta\Psi\right) + \Delta_R\Phi + q_n = 0,\\
&-D\Delta\Psi = \Gamma h\left(w - \Psi\right),
\end{aligned} \tag{20}$$

where the differential operators Δ and Δ_R are

$$\Delta w = \frac{1}{B}\frac{d}{ds}\left(B\frac{dw}{ds}\right) - \frac{m^2 w}{B^2}, \qquad \Delta_R w = \frac{1}{B}\frac{d}{ds}\left(\frac{B}{R_2}\frac{dw}{ds}\right) - \frac{m^2 w}{B^2 R_1}, \tag{21}$$

System (19), (20) consist of a separate equation (19) of the 2d order for function Θ and of three equations of the 8th order for functions $w,\ \Phi,\ \Psi$.

At the asymptotic analysis we seek the solution of system (20) in the form

$$w(s) = w_0 e^{\int p/R ds}. \tag{22}$$

In the same form we seek the rest unknown functions. After substituting (22) in (20) and omitting some small terms we get for the value p the characteristic equation of the 8th order

$$\mu_0^4(p^2-\rho^2)^4 + \left(1 - \eta_0(p^2-\rho^2)\right)(p^2 - k_1\rho^2)^2 = 0, \tag{23}$$

where $\eta_0 \sim h_*^{2-\delta}$ and $\mu_0 \sim h_*^{1/2}$ are the same as in (14) and

$$R = R_2, \quad k_1 = \frac{R}{R_1}, \quad \rho = \frac{mR}{B} \sim h_*^{-t_2} \quad t_2 < 1. \tag{24}$$

We seek the roots of equation (23) in the form

$$p = p_0 h_*^{-t_1}, \quad p_0 \sim 1 \quad (h_* \to 0) \tag{25}$$

where t_1 is the interesting for us index of variation of solution in the direction s. Equation (23) contains the parameters $\mu_0,\ \eta_0,\ \rho$ the orders of which are expressed through the main small parameter h_*. We suppose that $k_1 = O(1)$ and study two cases: $k_1 \sim 1$ and $k_1 = 0$. The case $k_1 = 1$ corresponds to the spherical shell, and the case $k_1 = 0$ corresponds to the cylindrical or to the conic shell. By using Newton's diagram we find the indexes of variation t_1 which are different in the

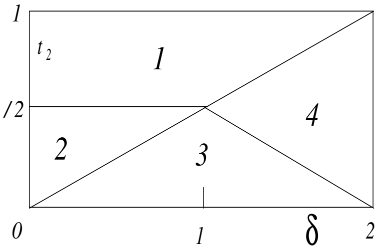

Fig. 3 The sub-domains with the various expressions for t_1.

Table 1 The indexes of variation t_1 of solutions in the s-direction.

1	t_2	8				
2	1/2	4			t_2 $at\ k_1 \neq 0$ $2t_2 - 1/2$ $at\ k_1 = 0$	4
3	$\delta/2$	2	t_2	6		
4	$\delta/2$	2	$1 - \delta/2$	2	t_2 $at\ k_1 \neq 0$ $2t_2 - 1/2$ $at\ k_1 = 0$	4

various parts of the domain $\{\delta,\, t_2\}$ $(0 \leq \delta < 2,\ 0 \leq t_2 < 1)$. In Fig. 3 the sub-domains with the different expressions for t_1 are shown and in Table 1 the values t_1 and the numbers of roots of equation (23) with such t_1 are given.

In the first column of Table 1 the number of the sub-domain from Fig. 2 is given. In the sub-domains *1* and *2* where $2t > \delta$ the shear influence is not essential. And in the sub-domains *3* and *4* the orders of some or of all roots of equation (23) essentially depend on the shear.

This asymptotic analysis is made for a static problem. The similar analysis may be fulfilled also for linear problems of free vibrations and of buckling. The stress-strain state is separated into the internal state for which the imaginary roots of equation (23) correspond and into the edge effect state for which the roots with the non-zero real parts correspond.

The analytical estimation of the errors of the 2D models by the comparison with the 3D solution is a very complex problem. Especially it is difficult to solve this problem near the shell edges where stress-strain state is essentially 3D [3], [4].

In the next sections for 2D theories the error of the internal state is investigated analytically and numerically for plates under the double periodic load.

3 The 3D equations for plates and their simplification

The equilibrium equations for the homogeneous plate are

$$\begin{aligned}
&\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0, \qquad i,j = 1,2,3, \\
&\sigma_{ii} = E_{ii}\varepsilon_{ii} + E_{ij}\varepsilon_{jj} + E_{ik}\varepsilon_{kk}, \quad i \neq j \neq k, \qquad \sigma_{ij} = G_{ij}\varepsilon_{ij}, \quad i \neq j, \\
&\varepsilon_{ii} = \frac{\partial u_i}{\partial x_i}, \qquad \varepsilon_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}, \quad i \neq j.
\end{aligned} \tag{26}$$

Here $x_1,\, x_2,\, x_3$ are the Cartesian co-ordinates in the tangential ($x_1,\, x_2$) and in the normal ($x_3 = z$) directions, u_i are the corresponding displacements, ε_{ij} are the strains, σ_{ij} are the stresses, and f_i are the external forces densities.

The elastic constants $E_{ij} = E_{ji},\ G_{ij} = G_{ji}$ for isotropic material depend on two elastic module (the Young's modulus E and the Poisson's ratio ν)

$$E_{ii} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}, \quad E_{ij} = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad G_{ij} = G = \frac{E}{2(1+\nu)}, \quad i \neq j. \tag{27}$$

For the transversal isotropic material which we study in details there are five independent elastic module ($E,\, E',\, G_{13},\, \nu,\, \nu'$) and the coefficients in (26) are [15]

$$\begin{aligned}
&E_{11} = E_{22} = \frac{E(1-\hat\nu^2)}{(1+\nu)(1-\nu-2\hat\nu^2)}, \quad E_{12} = \frac{E(\nu+\hat\nu^2)}{(1+\nu)(1-\nu-2\hat\nu^2)}, \\
&E_{13} = E_{23} = \frac{E\nu'(1+\nu)}{(1+\nu)(1-\nu-2\hat\nu^2)}, \quad E_{33} = \frac{E'(1-\nu)}{1-\nu-2\hat\nu^2}. \\
&G_{12} = G = \frac{E}{2(1+\nu)}, \quad G_{13} = G_{23}, \quad \hat\nu^2 = (\nu')^2\frac{E}{E'} < \frac{1-\nu}{2}.
\end{aligned} \tag{28}$$

We seek the double periodic solution of equations (26) in the form

$$
\begin{aligned}
u_1(x_1,\, x_2,\, z) &= u_1(z)\cos(r_1x_1)\sin(r_2x_2),\\
u_2(x_1,\, x_2,\, z) &= u_2(z)\sin(r_1x_1)\cos(r_2x_2),\\
u_3(x_1,\, x_2,\, z) &= u_3(z)\sin(r_1x_1)\sin(r_2x_2),
\end{aligned}
\tag{29}
$$

with the wave numbers r_1 and r_2. The corresponding stresses are

$$
\begin{aligned}
\sigma_{11} &= \sigma_{11}(z)\sin(r_1x_1)\sin(r_2x_2), & \sigma_{12} &= \sigma_{12}(z)\cos(r_1x_1)\cos(r_2x_2),\\
\sigma_{22} &= \sigma_{22}(z)\sin(r_1x_1)\sin(r_2x_2), & \sigma_{13} &= \sigma_{13}(z)\cos(r_1x_1)\sin(r_2x_2),\\
\sigma_{33} &= \sigma_{33}(z)\sin(r_1x_1)\sin(r_2x_2), & \sigma_{23} &= \sigma_{23}(z)\sin(r_1x_1)\cos(r_2x_2).
\end{aligned}
\tag{30}
$$

We study two test problems for which relations (29), (30) satisfy to equations (26). The first of them is the static one and it describes the external periodic compression with

$$
\begin{aligned}
&f_i = 0, \qquad \sigma_{i3}(-h/2) = 0,\ i = 1,2,3;\\
&\sigma_{i3}(h/2) = 0,\ i = 1,2, \quad \sigma_{33}(h/2) = p\sin(r_1x_1)\sin(r_2x_2),
\end{aligned}
\tag{31}
$$

where h is the plate thickness.

In the second problem we study the free vibrations with the frequency ω

$$
f_i = \rho\omega^2 u_i, \qquad \sigma_{i3}(\pm h/2) = 0, \quad i = 1,2,3,
\tag{32}
$$

where ρ is the material density.

In the both cases we ignore the boundary conditions at the plate edges or we suppose that the plate is infinite in the tangential directions. Nevertheless this approach is exact if the plate is of the form of parallelepiped with the special boundary conditions at the lateral edges.

System (26) contains 6 main unknown functions $u_i(z),\ \sigma_{i3}(z),\ i = 1,2,3$. The rest unknown functions are to be expressed by the main ones. For the case of vibrations system (26) may be re-written in the form (at the static case we put $\omega = 0$)

$$
\begin{aligned}
&\sigma_{13}' - (E_{11}r_1^2 + Gr_2^2)u_1 - (E_{12}+G)r_1r_2u_2 + E_{13}r_1u_3' + \rho\omega^2u_1 = 0,\\
&\sigma_{23}' - (E_{11}r_2^2 + Gr_1^2)u_2 - (E_{12}+G)r_1r_2u_1 + E_{13}r_2u_3' + \rho\omega^2u_2 = 0,\\
&\sigma_{33}' - G_{13}(r_1u_1' + r_2u_2') - G_{13}r^2u_3 + \rho\omega^2u_3 = 0,\\
&\sigma_{13} = G_{13}(u_1' + r_1u_3), \qquad \sigma_{23} = G_{13}(u_2' + r_2u_3),\\
&\sigma_{33} = -E_{13}(r_1u_1 + r_2u_2) + E_{33}u_3',
\end{aligned}
\tag{33}
$$

with $r^2 = r_1^2 + r_2^2,\ (\ )' = \partial/\partial x_3$.

The system (33) of the 6th order may be separated into 2 independent problems of 2nd and of 4th orders after introducing instead of u_1, u_2, σ_{13}, σ_{23} the unknown variables

$$
u = \frac{r_1u_1 + r_2u_2}{r}, \quad v = \frac{r_2u_1 - r_1u_2}{r}, \quad \sigma = \frac{r_1\sigma_{13} + r_2\sigma_{23}}{r}, \quad \tau = \frac{r_2\sigma_{13} - r_1\sigma_{23}}{r}
\tag{34}
$$

and after using the relation $E_{11} = E_{12} + 2G$.

The first of them consists of equations

$$
\tau' - r^2Gv + \rho\omega^2v = 0, \qquad G_{13}v' = \tau
\tag{35}
$$

and describes the curling motion and the second problem is

$$
\begin{aligned}
&\sigma' - E_{11}r^2u + E_{13}ru_3' + \rho\omega^2u = 0, && \sigma = G_{13}(u' + ru_3),\\
&\sigma_{33}' - r\sigma + \rho\omega^2u_3 = 0, && \sigma_{33} = E_{33}u_3' - E_{13}ru,
\end{aligned}
\tag{36}
$$

We mark that systems (35) and (36) contain the wave parameter r and do not depend on r_1 and r_2 separately.

4 The asymptotic solutions and the comparison with the 2D solutions

As a rule solutions of system (35) are used to satisfy the boundary conditions [7], [9] and system (36) describes the internal stress state of plate. We are bounded with the analysis of system (36). To fulfill the asymptotic analysis of system (36) we introduce the designations

$$z = h\hat{z}, \quad u = h\hat{u}, \quad u_3 = h\hat{u}_3, \quad \lambda = \frac{h^2\rho\omega^2}{E_0}, \quad \mu = rh \tag{37}$$

and then omit the symbol $\hat{}$. System (36) may be written as

$$\begin{aligned} u_3' &= \mu\frac{E_{13}u}{E_{33}} + \frac{\sigma_{33}}{E_{33}}, \\ u' &= -\mu u_3 + \frac{\sigma}{G_{13}}, \\ \sigma' &= \mu^2 E_0 u - \mu\frac{E_{13}\sigma_{33}}{E_{33}} - E_0\lambda u, \\ \sigma_{33}' &= \mu\sigma - E_0\lambda u_3 \end{aligned} \tag{38}$$

where

$$E_0 = \frac{E_{11}E_{33} - E_{13}^2}{E_{33}} = \frac{E}{1-\nu^2} \tag{39}$$

and this equality is valid as for the isotropic material (27) so for the transversal isotropic material (28).

If the wave length $L = 2\pi/r$ is much larger than the plate thickness h then μ is the small parameter which we use for the asymptotic expansions. Without loss of generality we put $u_{33} \sim 1$. Let us assume at first that all elastic module are of the same order

$$\{E_{11},\ E_{13},\ E_{33},\ G_{13}\} \sim E. \tag{40}$$

Then in the static case ($\lambda = 0$) from system (38) the following estimations may be delivered

$$u \sim \mu, \qquad \sigma \sim E\mu^3, \qquad \sigma_{33} \sim E\mu^4. \tag{41}$$

For the low-frequency vibrations with $\lambda \sim \mu^4$ the same estimations (41) are valid.

We seek the formal asymptotic expansions in powers of μ for system (38) solutions for the static problem (31) and for the vibration problem (32).

4.1 The static problem

In this case we seek the solution under conditions

$$\lambda = 0, \quad u_3(0) = 1, \quad \sigma(\pm 1/2) = 0, \quad \sigma_{33}(-1/2) = 0. \tag{42}$$

The right sides of equations (38) are small and the following solutions may be found by the simple iterations beginning from $u_3 = 1$, $u = \sigma = \sigma_{33} = 0$

$$\begin{aligned} u_3 &= 1 - \mu^2\frac{E_{13}z^2}{2E_{33}} + O(\mu^4), \\ u &= -\mu z + \mu^3\left(\frac{E_{13}(1+4z^3)}{24E_{33}} + \frac{E_0 z(3-4z^2)}{24G_{13}}\right) + O(\mu^5), \\ \sigma &= \mu^3\frac{E_0(1-4z^2)}{8} + \mu^5 E_0(1-4z^2)\left(\frac{E_{13}(1-2z^2)}{96E_{33}} - \frac{E_0(5-4z^2)}{128G_{13}}\right) + O(E\mu^7), \\ \sigma_{33} &= \mu^4 E_0\frac{(1+2z)^2(1-z)}{24} + \\ &+\mu^6(1+2z)^2 E_0\left(\frac{E_{13}(3-2z-4z^3+4z^3)}{960E_{33}} - \frac{E_0(8-7z-4z^2+4z^3)}{1920G_{13}}\right) + O(E\mu^8). \end{aligned} \tag{43}$$

To satisfy the last condition (31) we ought to scale solution (43). Then we finally find the deflection of the plate mid-plane u_{30}

$$u_{30} = \frac{pu_3(0)}{\sigma_{33}(1/2)}, \qquad \sigma_{33}(1/2) = \mu^4 \frac{E_0}{12} + \mu^6 E_0 \left(\frac{E_{13}}{160E_{33}} - \frac{E_0}{120G_{13}} \right) + O(\mu^8). \tag{44}$$

By using designations (37) for the transversal deflection the 2D KL theory (u_3^K) and the TR theory (u_3^T) give

$$u_3^K = \frac{12p}{E_0\mu^4}, \qquad u_3^T = \frac{12p}{E_0\mu^4} + \frac{p}{\Gamma\mu^2}, \tag{45}$$

where $\Gamma = \kappa G_{13}$ is the equivalent shear modulus and the various values are used for the dimensionless coefficient κ. We rewrite relation (44) in the form

$$u_{30} = \frac{12p}{E_0\mu^4} + \frac{6p}{5G_{13}\mu^2} - \frac{9pE_{13}}{10E_0E_{33}\mu^2} + O(\mu^0). \tag{46}$$

The comparison of relations (45) and (46) shows that the KL theory gives the first asymptotic approximation at $\mu \to 0$ of the 3D solution (46). If $\kappa = 5/6$ the TR theory gives one of two summands of the second order in (46). The other term of the second order in (46) is connected with the plate normal deformation and this effect is not described as by the KL theory so by the TR theory. In the book [15] by using some hypotheses about the distribution of stresses and strains in the normal direction the more complex 2D theory is constructed. In partial this theory takes into consideration the deformations of the normal element. For shells this theory has the 14th order while the KL theory is of the 8th order and the TR theory is of the 10th order.

4.2 The vibration problem

We seek the solution of system (38) under conditions

$$\sigma(\pm 1/2) = \sigma_{33}(\pm 1/2) = 0. \tag{47}$$

The first terms of the asymptotic expansions of solution are

$$\begin{aligned} u_3 &= 1 - \mu^2 \frac{E_{13}z^2}{2E_{33}} + O(\mu^4), \\ u &= -\mu z + \mu^3 \left(\frac{E_{13}z^3}{6E_{33}} + \frac{E_0(3z - 4z^3)}{24G_{13}} \right) + O(\mu^5). \end{aligned} \tag{48}$$

The lowest eigen-value λ is to be found from the compatibility condition for the last equation of system (38)

$$\lambda + \frac{\mu^2\lambda(2E_{33} - 5E_{13})}{24E_{33}} - \frac{\lambda^2 E_0}{24E_{33}} = \frac{\mu^4}{12} - \frac{E_0\mu^6}{120G_{13}} + O(\mu^8). \tag{49}$$

Taking into account that $\lambda = O(\mu^4)$ we hold in equation (49) all terms of the orders $O(\mu^4)$ and $O(\mu^6)$ and the summand containing λ^2.

The KL theory gives in our designations the relation

$$\lambda^K = \frac{\mu^4}{12}, \tag{50}$$

which corresponds to the main terms of equation (49).

The TR theory gives the following equation for λ^T which includes the rotation inertia

$$\lambda^T + \frac{E_0\mu^2\lambda^T}{12\Gamma} + \frac{\mu^2\lambda^T}{12} - \frac{(\lambda^T)^2 E_0}{12\Gamma} = \frac{\mu^4}{12}. \tag{51}$$

If in the second term of the left side (51) we put $\lambda^T = \mu^4/12$ and $\Gamma = 5G_{13}/6$ then it coincides with the corresponding term in the right side of (49). The rest terms in (51) including the term with $(\lambda^T)^2$ which describes the rotation inertia differ from the corresponding terms in (49).

These two test problems allow us to make the conclusion that the KL theory is asymptotically correct because it correctly gives the main terms of the asymptotic expansion of the exact 3D solution. For the case $G_{13} \sim E$ the TR theory is asymptotically incorrect because it correctly describes only the part of the terms of the second order in the exact solution. But if $G_{13} << E$ then the terms containing G_{13} and describing the transversal shear becomes much larger than the rest terms of the second order and the application of this theory really gives the correction compared with the KL theory.

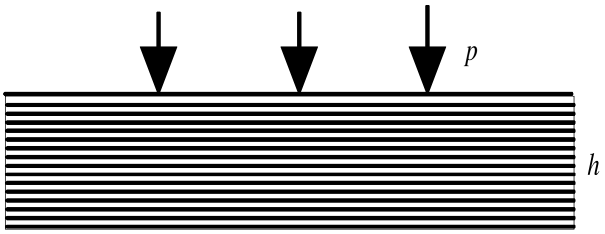

Fig. 4 The multi-layered plate.

5 The multi-layered plate

As an example we study the transversal isotropic plate which is a limit of the multi-layered plate consisting of the alternating hard and soft isotropic layers when the thickness of layers goes to zero and their number goes to infinity.

Let the parameters of isotropic layers are

$$E_k,\, \nu_k,\, h_k, \qquad k = 1, 2. \tag{52}$$

Assuming that the strains $\varepsilon_{11},\, \varepsilon_{12},\, \varepsilon_{22}$ and the stresses $\sigma_{13},\, \sigma_{23},\, \sigma_{33}$ are continuous we get the elastic module $E_{ij},\, G_{ij}$ in (26) for the equivalent transversal isotropic layer

$$\begin{aligned}
&E_{ij} = \frac{h_1 E_{ij}^{(1)} + h_2 E_{ij}^{(2)}}{h_1 + h_2}, \quad i,j = 1,2; \qquad G_{12} = \frac{h_1 G^{(1)} + h_2 G^{(2)}}{h_1 + h_2},\\
&E_{13} = E_{23} = \frac{\nu_1 h_1/(1-\nu_1) + \nu_2 h_2/(1-\nu_2)}{h_1/E_{11}^{(1)} + h_2/E_{11}^{(2)}}, \quad E_{33} = \frac{h_1 + h_2}{h_1/E_{11}^{(1)} + h_2/E_{11}^{(2)}}\\
&G_{13} = G_{23} = \frac{h_1 + h_2}{h_1/G^{(1)} + h_2/G^{(2)}},
\end{aligned} \tag{53}$$

where $E_{1j}^{(k)},\, G^{(k)}$ for the layer k are given in (27).

We study the static problem (31) for the plate with N layers and seek the deflection $u_{33}(0)$. We compare four solutions of this problem
(KL) — the KL approximation,
(TR) — the TR approximation,
(tr) — the exact 3D solution for the equivalent transversal isotropic plate,
(mul) — the exact 3D solution for the multi-layered plate.

Solutions of the problems (KL) and (TR) are given by relations where the equivalent elastic module (53) are used. Solutions of the problems (tr) and (mul) are obtained numerically from the system (36) with the boundary conditions

$$\sigma(\pm h/2) = 0, \quad \sigma_{33}(-h/2) = 0, \quad \sigma_{33}(h/2) = p. \tag{54}$$

In this problem the solution of system (35) is equal to zero. For the problem (tr) we take the equivalent module (53), and for the problem (mul) the elastic module are the piecewise constant ones according to (52).

For numerical example we take

$$E_1/E_2 = 100, \quad h_1 = h_2, \quad \nu_1 = 0.3, \quad \nu_2 = 0.45. \tag{55}$$

The results of calculations are contained in Table 2. We are interesting only with the relations between results obtained by various theories. In the columns 2 and 3 the values u_3^K/u_{30} and u_3^T/u_{30} are presented where u_3^K and u_3^T are calculated by relations (45) and u_{30} is the deflection which is found numerically for the equivalent transversal isotropic homogeneous plate. In the last four columns the values u_3^{mul}/u_{30} for four various numbers of layers are given. Here again the values u_3^{mul} are found numerically for the multi-layered plate. Nine various values $h/L = 2\pi\mu$ are examined (see column 1), where L is the length of wave.

For this plate the value $G_{13}/E_0 = 0.012$ is small enough. That is why the value u_3^T/u_{30} is close to 1 in the wide limits for h/L. The coincidence between the transversal isotropic plate and the multi-layered plate becomes better if the number N of layers is larger. The full coincidence is impossible because the difference between the numbers of hard layers and of soft layer is equal to unit.

6 The extremely small transversal shear modulus

If the transversal shear modulus is very small then some peculiarities in the stress state appear. In the section 2 the edge effect degeneration is marked. Here we discuss one of the shell buckling problems.

We study the buckling under axial compression of the circular cylindrical shell of the radius R made of the transversal isotropic material. By using the TR theory the problem is reduced to system (20) in which

$$q_n = T_0 \frac{\partial^2 w}{\partial s^2}, \tag{56}$$

where $T_0 < 0$ is the initial stress-resultant. We seek the solution of this system in the form

$$w(s, \varphi) = w_0 \sin(rs) \sin(m\varphi). \tag{57}$$

After the substitution of the expression (57) and of the similar expressions for Φ and Ψ in system (20) we get

$$-T_0 = \frac{D(r^2 + m^2)^4 + (Eh/R^2) r^4 (1 + (D/\Gamma h)(r^2 + m^2))}{r^2 (r^2 + m^2)^2 (1 + (D/\Gamma h)(r^2 + m^2))}. \tag{58}$$

The critical value T_0 is equal to the minimum of the right side in (58) with respect to the wave numbers r and m. At first we establish that if $\Gamma < \infty$ then the minimum is obtained at $m = 0$ therefore the buckling mode is axisymmetrical. We mark that the KL theory leads to the various buckling modes as axisymmetrical so non-symmetrical with the same critical value T_0 [21]. After minimizing (58) with respect to r we get

$$-T_0 = \frac{Eh^2}{R\sqrt{12(1-\nu^2)}} f(\lambda), \qquad f(\lambda) = \begin{cases} 2 - \lambda, & \lambda \le 1, \\ 1/\lambda, & \lambda \ge 1, \end{cases} \qquad \lambda = \frac{Eh}{\Gamma R \sqrt{12(1-\nu^2)}}. \tag{59}$$

The function $f(\lambda)$ describes the effect of the critical load decrease due to the shear. At $\lambda = 0$ we get $f(0) = 2$. In this case relation (59) coincides with the classical Lorentz–Timoshenko formula [22], [23].

At $\lambda > 0$ the function $f(\lambda)$ is described by two various analytical expressions. If $\lambda < 1$ then $f(\lambda) = 2 - \lambda$ and the critical value is not more than twice smaller than the classical one. In this case the minimum is reached at some finite wave number r. If $\lambda > 1$ the relation (59) accepts the form

$$-T_0 = \Gamma h \tag{60}$$

and the critical load depends only on the shear. The value (60) is obtained from (58) as a limit at $r \to \infty$. But we remember that the 2D shell theories are acceptable if $h/L = 2\pi r h << 1$. Therefore the result (60) is doubtful. To partly explane relation (60) we mark that the initially compressed material is unstable if the compression is large enough. In partial the transversal isotropic material becomes instable if

$$-\sigma^0 \ge G_{13}, \tag{61}$$

where σ^0 is the initial stress in the plane x_1, x_2, and G_{13} is the tranversal shear elastic modulus. If we mark that in the shell problem

$$\sigma^0 = \frac{T_0}{h} = -\frac{5}{6} G_{13} \tag{62}$$

then it will be clear that in the case $\lambda \ge 1$ (or may be $\lambda \ge 5/6$) we have not the shell buckling but the loss stability of material.

These examples show that for the extremely small shear module the special investigations are necessary.

Table 2 Comparison of the deflections obtained by various theories.

h/L	u_3^K/u_{30}	u_3^T/u_{30}	$N = 11$	$N = 21$	$N = 51$	$N = 101$
0.5	0.015	1.178	0.614	0.856	0.964	0.986
0.2	0.078	1.077	0.886	0.947	0.978	0.989
0.1	0.244	1.028	0.905	0.943	0.974	0.986
0.05	0.558	1.007	0.862	0.917	0.962	0.980
0.02	0.887	1.001	0.809	0.888	0.950	0.974
0.01	0.969	1.000	0.796	0.880	0.947	0.972
0.005	0.992	1.000	0.792	0.878	0.946	0.972
0.002	0.999	1.000	0.791	0.878	0.946	0.972
0.001	1.000	1.000	0.791	0.878	0.946	0.972

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